# Black hole thermodynamics and Hawking radiation

(Just an overview by a non-expert on the field!)

Renato Fonseca - 26 March 2018 – for the Journal Club

# Thermodynamics meets GR

- Research in the 70's convincingly brought together two very different areas of physics: thermodynamics and General Relativity
- People realized that black holes (BHs) followed some laws similar to the ones observed in thermodynamics

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

$$dU = \delta Q + \delta W = TdS - pdV + \Omega dJ + \Phi dQ + \cdots$$

$$\text{Entropy } (S) \propto \text{BH area } (A)$$
$$\text{Temperature } (T) \propto \text{BH surface gravity } (\kappa)$$

- It was then possible to associate a temperature to BHs. But was this just a coincidence?
- No. Hawkings (1975) showed using QFT in curved space that BHs from gravitational collapse radiate as a black body with a certain temperature T

# Black Holes

# Schwarzschild metric (for BHs with no spin nor electric charge)

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- All coordinates (t,r,theta,phi) are what you think they are far from the central mass M
- Something funny seems to happen for r=2M. But … locally there is nothing special there (only at r=0):

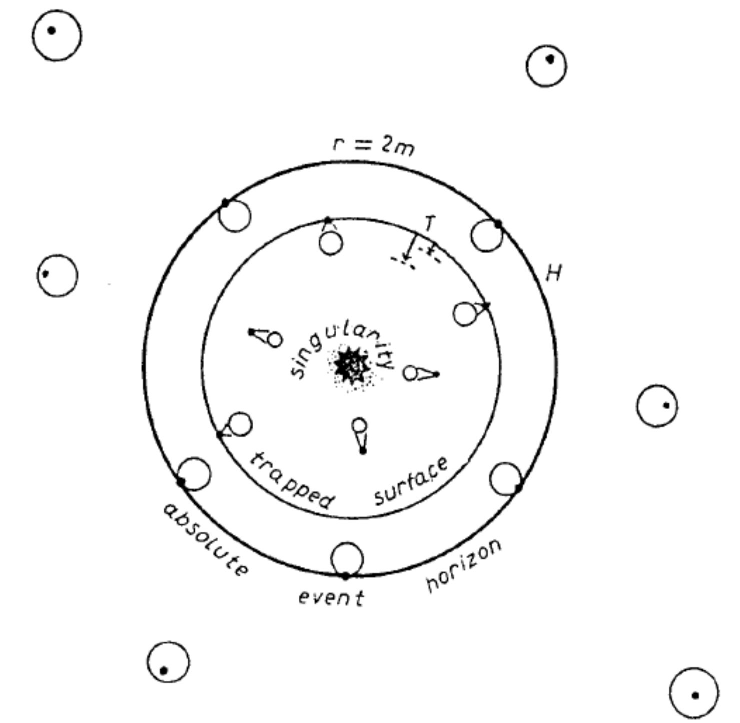
$$R_{abcd}R^{abcd} = \frac{48M^2}{r^6}$$

r=0: real/intrinsic singularity
r=2M: apparent singularity (can be removed with other coordinates)

Important caveat: the Schwarzschild solution is NOT what is called *maximal*. With coordinates change we can get the Kruskal solution, which is. "*Maximal*"= ability to continue geodesics until infinity or an intrinsic singularity

# Schwarzschild metric (for BHs with no spin nor electric charge)

- r=2M (*event horizon*) is not special for its LOCAL properties (curvature, etc) but rather for its GLOBAL properties: r<=2M are *closed trapped surfaces* which cannot communicate with the outside world [dr/dt=0 at r=2M even for light]




(Penrose, 1969)

Fun fact: for null geodesics (=light) we see that

$$\frac{dt}{dr} = \frac{\frac{dt}{du}}{\frac{dr}{du}} = \pm\frac{1}{1-\frac{2M}{r}} = \pm\frac{r}{r-2M}$$

+/- = light going out/in

One can even integrate this:

$$t(r) = \pm\left[r + 2M\log|r-2M|\right] + \mathrm{const.}$$

t=infinite for even light to fall into the BH! This is what an observatory at infinity sees …

# Schwarzschild metric (digression)

- But the object itself does fall into the BH. And very fast… the reason is there is a mismatch between t and local time tau (which is infinitely bad for r=2M):

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

- For an object starting at r0 for tau0 …

$$\tau - \tau_0 = \frac{2}{3\sqrt{2M}}\left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}\right)$$

Example: if the Sun turned into a black hole, an object at the event horizon would hit the central singularity in 6.55 microseconds!

$$M_{Sol} = 1,99 \times 10^{30}\ \mathrm{kg},\ r_0 = 2M_{Sol} = 2,95\ \mathrm{km} \longrightarrow \tau = 6,55\ \mu\mathrm{s}$$

# Other BHs

- BHs can have mass M, angular momentum J, electric charge Q:
  - (M,Q): Reissner-Nordstrom solution
  - (M,J): Kerr solution
  - (M,J,Q): Kerr-Newman solution
- Probably Q~0 in all realistic cases, so the Kerr solution is probably the most general one for realistic scenarios. In the Boyer-Lindquist coordinates:

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - adt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

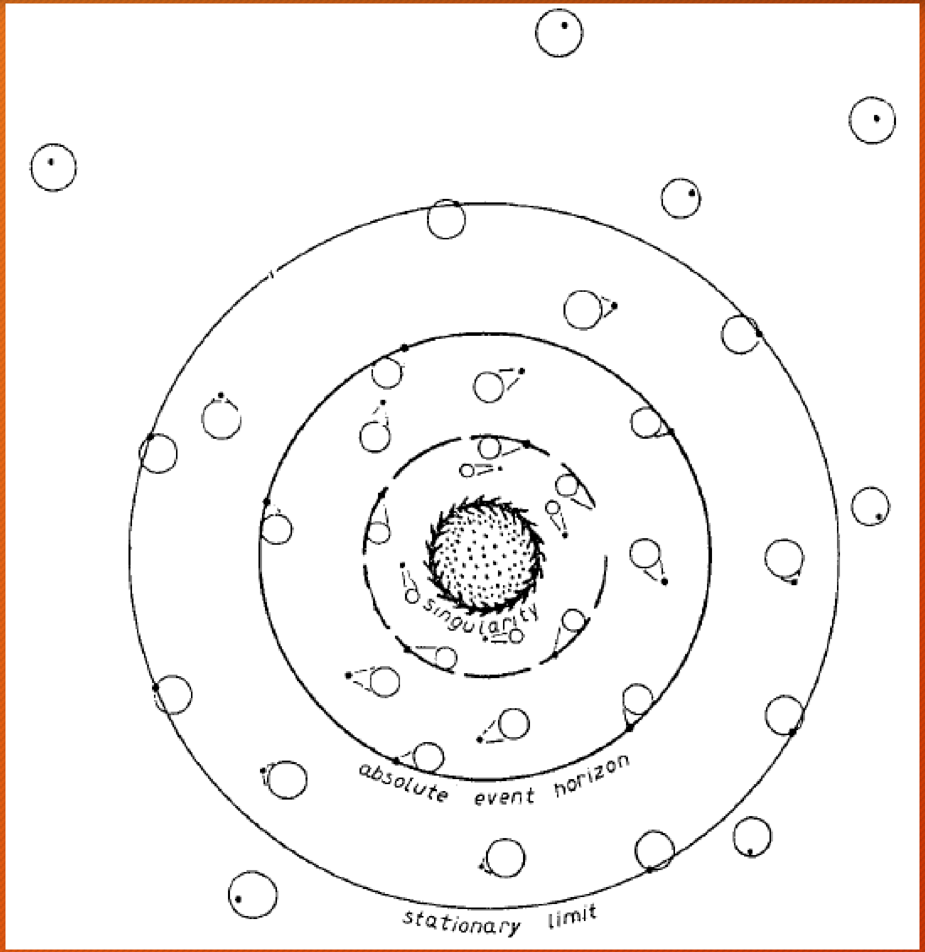
BH parameters:
M and a=J/M

$$\begin{aligned}\rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + a^2\end{aligned}$$

$$\begin{aligned}x &= r\sin\theta\cos\phi + a\sin\theta\sin\phi \\ y &= r\sin\theta\sin\phi - a\sin\theta\cos\phi \\ z &= r\cos\theta\end{aligned}$$

t,x,y,z are the standard coordinates

# Kerr Black Hole (with angular momentum J)




(Penrose, 1969)

Equatorial cut of the BH. Note …

(1) - the stationarity below which d(phi)/dt!=0 due to *frame dragging*

(2) - An outer and an inner event horizons

(3) - A singularity with a ring shape

# BHs: should we trust this General Relativity prediction?

Yes. On to the news slide. Just kidding … let's see why BHs do probably exist

- At r=0 indeed we have an infinite density
- But at the event horizon, local quantities (densities, curvatures,…) are finite. And remember: <u>the defining property of a BH is the event horizon</u>
- BH average density is very high? Curvature is very high at the event horizon? Not necessarily!
- Penrose gave this example: bring together 10^11 stars like the Sun until they are almost touching. The system would form a BH! And the average density would be ~2g/m^3 (close to the one of air) and the curvature would be several orders of magnitude less that the one you are feeling right now due to the Earth's mass.
- Why does this happen? There is no BH density: call that r~M so large BH can be very "mild". If we trust GR, QFT on Earth then we should probably also trust them for these mass concentrations

# Thermodynamics of BHs

# Area and surface gravity of BHs
(entropy) (temperature)

Area (concept is obvious)

$$A = \int_0^{2\pi}\int_0^{\pi} |\sin\theta| \left(r_+^2 + a^2\right) d\theta d\phi$$

$$= 4\pi\left(2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}}\right)$$

Surface gravity (not so obvious)

$$\nabla^a\left(X^b X_b\right) \equiv -2\kappa X^a$$

X is a properly normalized Killing vector field (let's ignore the details and just keep a faint idea of what is going on…)

$$\kappa = \frac{1}{4M}$$ (Schwarzschild)

$$\kappa = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2}}$$ (Kerr-Newman)

# Laws of thermodynamics for BHs

**Black Holes and the Second Law** (*).

J. D. BEKENSTEIN (**)

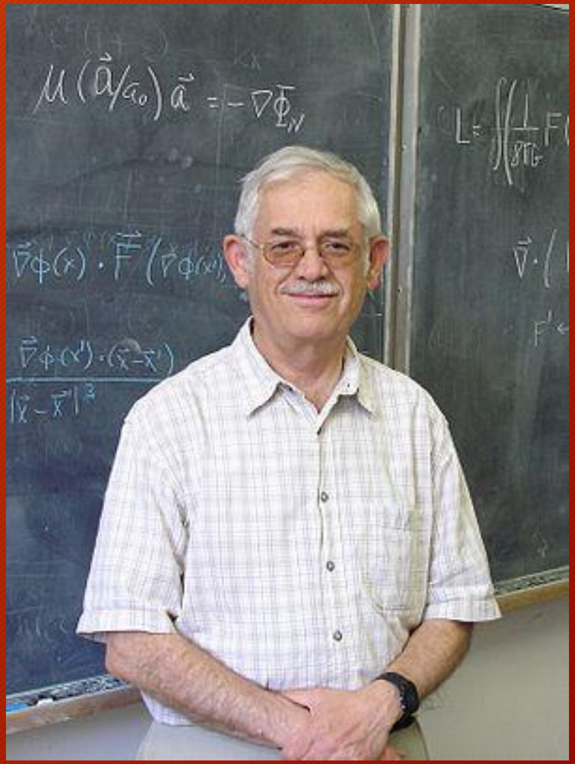
*Joseph Henry Laboratories, Princeton University - Princeton, N. J.*

(ricevuto il 22 Maggio 1972)

Black-hole physics seems to provide at least two ways in which the second law of thermodynamics may be transcended or violated:

*a*) Let an observer drop or lower a package of entropy into a black hole; the entropy of the exterior world decreases. Furthermore, from an exterior observer's point of view a black hole in equilibrium has only three degrees of freedom: mass, charge and angular momentum (1). Thus, once the black hole has settled down to equilibrium, there is no way for the observer to determine its interior entropy. Therefore, he cannot exclude the possibility that the total entropy of the universe may have decreased in the process. It is in this sense that the second law appears to be transcended (2).

*b*) A method for violating the second law has been proposed by GEROCH (3): By



1 May 1947 – 16 August 2015

# Laws of thermodynamics for BHs

PHYSICAL REVIEW D VOLUME 7, NUMBER 8 15 APRIL 1973

## Black Holes and Entropy*

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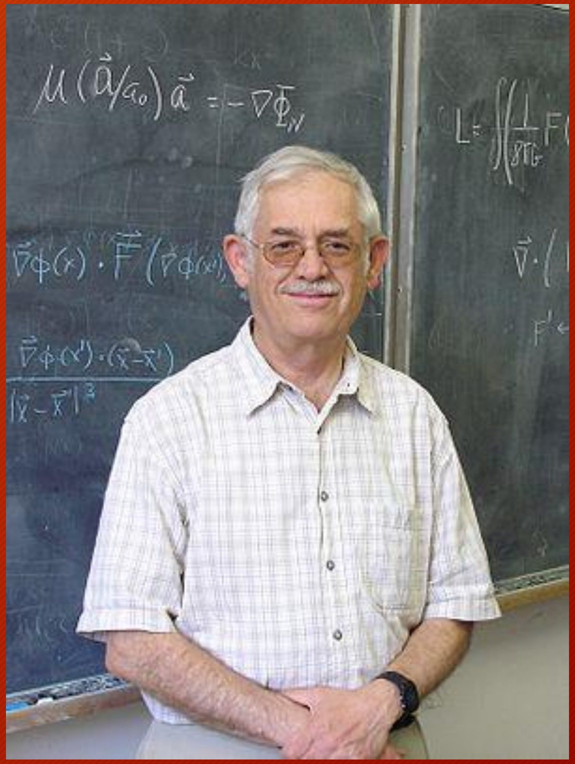
(Received 2 November 1972)

There are a number of similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black-hole area and of entropy: Both quantities tend to increase irreversibly. In this paper we make this similarity the basis of a thermodynamic approach to black-hole physics. After a brief review of the elements of the theory of information, we discuss black-hole physics from the point of view of information theory. We show that it is natural to introduce the concept of black-hole entropy as the measure of information about a black-hole interior which is inaccessible to an exterior observer. Considerations of simplicity and consistency, and dimensional arguments indicate that the black-hole entropy is equal to the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of order unity. A different approach making use of the specific properties of Kerr black holes and of concepts from information theory leads to the same conclusion, and suggests a definite value for the constant. The physical content of the concept of black-hole entropy derives from the following generalized version of the second law: When common entropy goes down a black hole, the common entropy in the black-hole exterior plus the black-hole entropy never decreases. The validity of this version of the second law is supported by an argument from information theory as well as by several examples.



1 May 1947 – 16 August 2015

# Laws of thermodynamics for BHs

Zeroth law: *If two systems are in thermal equilibrium with a third system, they are in thermal equilibrium with each other. This law helps define the concept of temperature*

(Wikipedia)

A BH will be in thermal equilibrium with itself only of the temperature (surface gravity) is constant. This is true. ✓

First law: *When energy passes, as work, as heat, or with matter, into or out from a system, the system's internal energy changes in accord with the law of conservation of energy*

(Wikipedia)

A BH's mass will change according to the extensive variables as expected if we <u>interpret A and kappa as being proportional to S (entropy) and T (temperature)</u>

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

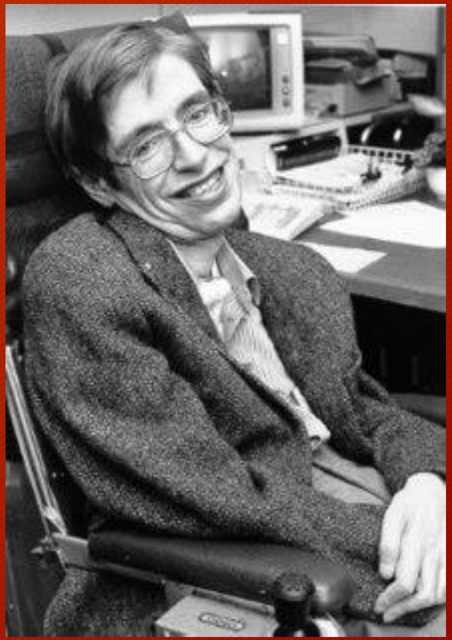
$$dU = \delta Q + \delta W = TdS - pdV + \Omega dJ + \Phi dQ + \cdots$$

✓

# Laws of thermodynamics for BHs

Second law: *In a natural thermodynamic process, the sum of the entropies of the interacting thermodynamic systems increases* (Wikipedia)

In 1972, Stephen Hawking showed that $dA>=0$: area of a black hole never decreases classically (it's mass can)



8 January 1942 – 14 March 2018

# Laws of thermodynamics for BHs

Black Holes in General Relativity

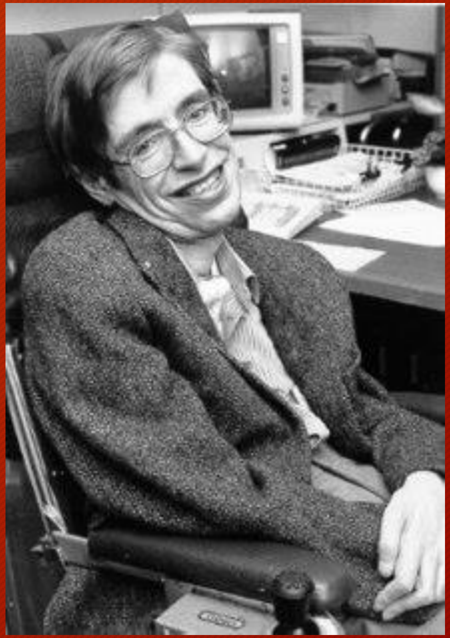
S. W. HAWKING

Institute of Theoretical Astronomy, University of Cambridge, Cambridge, England

Received October 15, 1971

In Section 2 a black hole is defined in terms of a event horizon and it is shown that the surface area of a black hole cannot decrease with time.

Since the generators of $\dot{J}^-(\mathscr{I}^+)$ have no future end points and have convergence $\varrho \leqq 0$, the surface area of $\partial\mathscr{B}_1(t)$ cannot decrease with $t$. If two black holes $\mathscr{B}_1(t_1)$ and $\mathscr{B}_2(t_1)$ on the surface $\mathscr{S}(t_1)$ merge to form a single black hole $\mathscr{B}_3(t_2)$ on a later surface $\mathscr{S}(t_2)$, then the area of $\partial\mathscr{B}_3(t_1)$ must be at least the sum of the areas of $\partial\mathscr{B}_1(t_1)$ and $\partial\mathscr{B}_2(t_1)$. In fact it must be strictly greater than this sum because $\partial\mathscr{B}_3(t_2)$ contains two



8 January 1942 – 14 March 2018

# Laws of thermodynamics for BHs

Third law: *Impossibility of reaching the absolute zero temperature*

For BHs, the surface gravity

$$\kappa = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2}}$$

can be 0 if

$$\frac{Q^2}{M^2} + \frac{J^2}{M^4} = 1$$

✓ ?

But if this happens, a zero temperature will be the least of our problems… In particular these extreme BHs would have no event horizon protecting the outside Universe from the singularity. The *cosmic censorship hypothesis* posits that such naked singularities do not exist.

# Boltzmann equation strikes back $S = k \log W$

What is entropy microscopically (at least in a normal setup)?
It measures the number of microstates W which correspond to a given macrostate (given by T,V,P,…)

Black holes have no hairs (or rather they have just 3: M, J, Q) so it makes sense that they have maximum entropy

**Classically**

Assume that the BH is made up of n particles, each with chi degrees of freedom

$$W \propto \chi^n$$

For n can be infinite if each particles has a tiny mass/energy (like a photon)

**Quantum mechanics**

Seems reasonable to require that the wavelength of the particles which form the BH is smaller than the BH

$$E = \frac{h}{\lambda} > \frac{h}{2M}$$

$$S = k \log \chi^n = \xi k \frac{M^2}{h} \qquad \xi = 2 \log \chi$$

Entropy is not infinite

# Where does this leave us?

$$S = \frac{\xi k}{16\pi h} A$$

$$T = \frac{M}{2S} = \frac{h}{2\xi k M} = \frac{2h}{\xi k}\kappa$$

Everything seems to point to the fact that <u>entropy is proportional to area</u>; <u>temperature is proportional to surface gravity</u>

But what is the constant of proportionality ?

$$T = \frac{\hbar}{2\pi k}\kappa$$

i.e. $\xi = 8\pi^2$

# Hawking radiation

## Particle Creation by Black Holes

S. W. Hawking

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Received April 12, 1975

**Abstract.** In the classical theory black holes can only absorb and not emit particles. However it is shown that quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with temperature $\frac{\hbar\kappa}{2\pi k} \approx 10^{-6}\left(\frac{M_\odot}{M}\right)$ °K where $\kappa$ is the surface gravity of the black hole. This thermal emission leads to a slow decrease in the mass of the black hole and to its eventual disappearance: any primordial black hole of mass less than about $10^{15}$ g would have evaporated by now. Although these quantum effects violate the classical law that the area of the event horizon of a black hole cannot decrease, there remains a Generalized Second Law: $S+\frac{1}{4}A$ never decreases where $S$ is the entropy of matter outside black holes and $A$ is the sum of the surface areas of the event horizons. This shows that gravitational collapse converts the baryons and leptons in the collapsing body into entropy. It is tempting to speculate that this might be the reason why the Universe contains so much entropy per baryon.

# But first … let's quickly see some applications

- Imagine a BH in a thermal bath at a temperature $T_B$. What do you think will happen? BH will grow forever right?
- Not necessarily!

$$F = M - T_B S \qquad dF = 0 \qquad d^2F > 0$$

**Equilibrium condition:** Helmholtz free energy F must be in a stable minimal

$$\alpha > 2\sqrt{3-\beta} - 3 \longleftarrow$$ **result**

$$\begin{aligned} J^2 &\equiv \alpha M^4 \\ Q^2 &\equiv \beta M^2 \end{aligned}$$

One can show that this is equivalent to having a thermal capacity (at constant J,Q) positive

$$C_{J,Q} \equiv \left(\frac{\partial M}{\partial T}\right)_{J,Q} > 0$$

(Note: for J=Q=0 this is not possible, but otherwise it is)

# But first … let's quickly see some applications

- Second case: put a BH is a box, and wait for it to fill the box with radiation. When is equilibrium reached?

$$S = \frac{1}{2}M^2 + \frac{4}{3}\left(aVM_r^3\right)^{\frac{1}{4}}$$

(add entropy of BH and radiation)

$dS = 0$

$$\begin{aligned} dS &= \frac{dS}{dM}dM \\ &= \left(\frac{\partial S}{\partial M} + \frac{\partial S}{\partial M_r}\frac{\partial M_r}{\partial M}\right)dM \\ &= \left[M - \left(\frac{aV}{M_r}\right)^{\frac{1}{4}}\right]dM \end{aligned}$$

$$aV = M^4 M_r$$

$d^2S < 0$

$$\begin{aligned} d^2S &=|_{eq.} \left[\frac{\partial^2 S}{\partial M^2} + 2\frac{\partial^2 S}{\partial M \partial M_r}\frac{\partial M_r}{\partial M} + \frac{\partial^2 S}{\partial M \partial M_r}\left(\frac{\partial M_r}{\partial M}\right)^2 + \frac{\partial^2 S}{\partial M_r^2}\frac{\partial M_r}{\partial M} + \frac{\partial S}{\partial M_r}\frac{\partial^2 M_r}{\partial M^2}\right]dM \\ &= \left[\frac{\partial^2 S}{\partial M^2} + \frac{\partial^2 S}{\partial M_r^2}\left(\frac{\partial M_r}{\partial M}\right)^2\right]dM \\ &= \left[1 - \frac{1}{4}\left(\frac{aV}{M_r}\right)^{\frac{1}{4}}\frac{1}{M_r}\right]dM \end{aligned}$$

$$M > 4M_r$$

i.e., BH must have 4/5 or more of the total energy

# But first … let’s quickly see some applications

ability, the volume $V$ of the box must be sufficien
ly small that the energy $E_1$ of the blackbody
gravitons is less than $\frac{1}{4}$ the mass of the black
hole. (Note that this result depends only on the

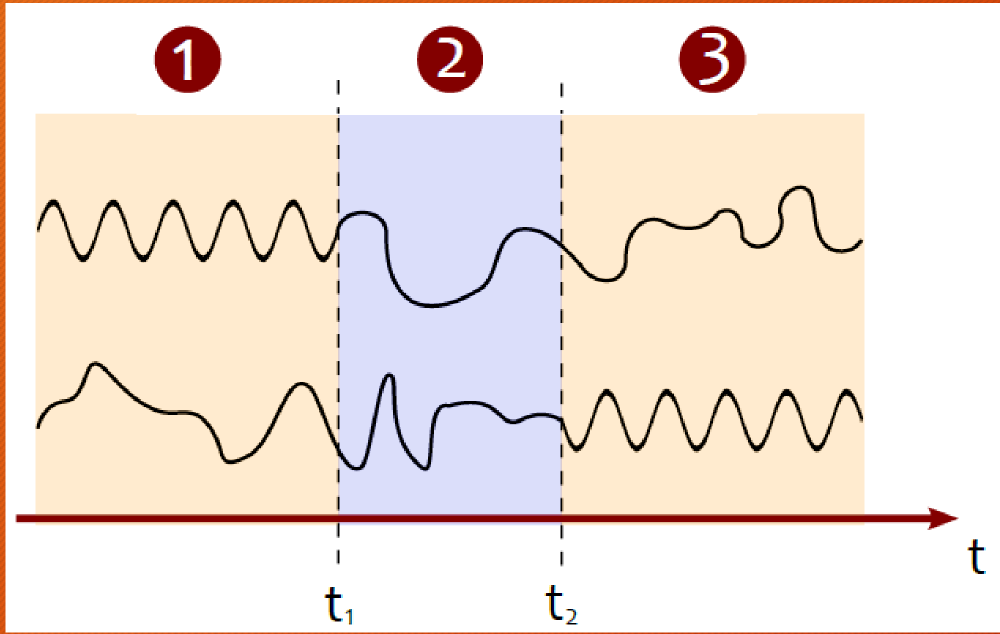
Hawking 1976

# Back to Hawking radiation … won’t say much

- Not all BHs radiate (for example, ethernal ones do not)
- Surprisingly, the BH radiation has its origin in the BH formation from gravitational collapse (but details of the collapse are irrelevant for distant observer)
- **What is going on?** BH are peculiar creatures of space-time. By looking at a BH, an observer is seeing details of the Universe before the BH was formed. In particular, the Universe’s past vacuum is seem not as a vacuum, but as a mixture of particles with a thermal spectrum, apparently coming from the BH

# The basic idea … pictorically

flat space | curved space | flat space



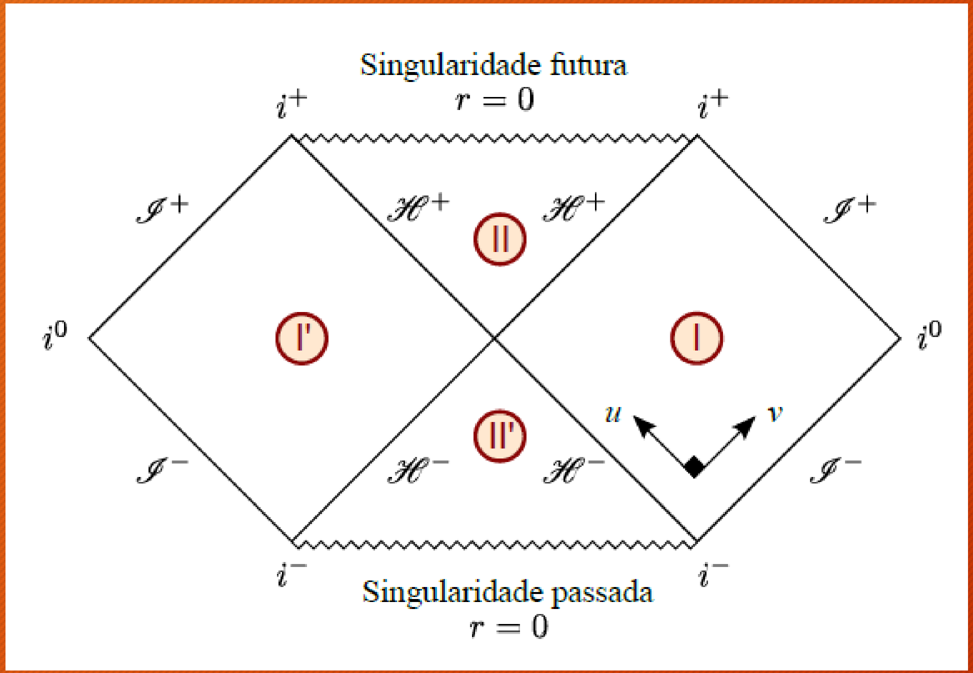

$$u_i\left(x^a\right) \equiv \propto \exp\left(ik_i^a x_a\right) \quad , k_i = \left(\omega_i, \overrightarrow{k_i}\right) \text{ e } \omega_i > 0$$

Plane waves with positive frequency in region 1 appear as a <u>mixture of plane waves</u> with positive and negative frequencies in region 3. And vice versa.

$$\begin{aligned} \phi_i &= \sum_j A_{ij} u_j + B_{ij} u_j^* \\ a_i^3 &= \sum_j A_{ji} a_j^1 + B_{ji}^* a_j^{1\dagger} \end{aligned}$$

$$\begin{aligned} \langle 0|_1 N_i^1 |0\rangle_1 &= \langle 0|_1 a_i^{1\dagger} a_i^1 |0\rangle_1 = 0 \\ \langle 0|_1 N_i^3 |0\rangle_1 &= \langle 0|_1 a_i^{3\dagger} a_i^3 |0\rangle_1 \\ &= B_{ji} B_{ki}^* \langle 0|_1 a_j^1 a_k^{1\dagger} |0\rangle_1 \\ &= B_{ji} B_{ki}^* \left(\delta_{jk} \langle 0|0\rangle_1 - \langle 0|_1 a_{1\,j}^{\dagger} a_{1\,k} |0\rangle_1\right) \\ &= B_{ji} B_{ji}^* \\ &= \left[B^\dagger B\right]_{ii} \end{aligned}$$

# The basic idea … pictorically




Penrose diagram for the Kruskal solution (maximal analytic solution of the Schwarzschild solution)

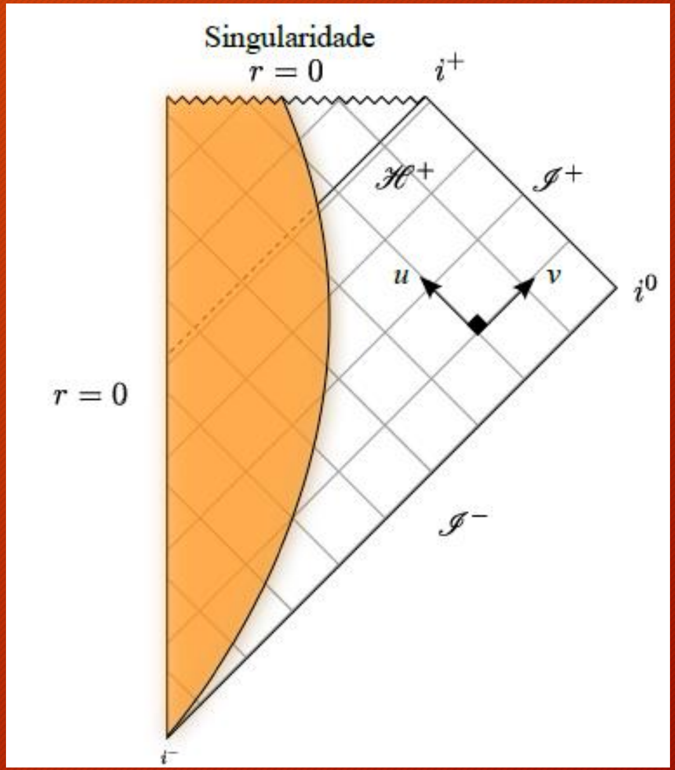



Diagram for BH from gravitational collapse

# The basic idea …

$$\Phi^{in}_{\omega lm} : \quad \Phi^{in-}_{\omega lm} = \frac{1}{\sqrt{2\pi\omega}} \frac{1}{r} h^{in}_{\omega}(r) \exp(i\omega v) Y^m_l(\theta, \phi)$$

One frequency mode in the distant past

$$\Phi^{out}_{\omega lm} : \quad \Phi^{out+}_{\omega lm} = \frac{1}{\sqrt{2\pi\omega}} \frac{1}{r} h^{out}_{\omega}(r) \exp(i\omega u) Y^m_l(\theta, \phi)$$

One frequency mode in the distant future

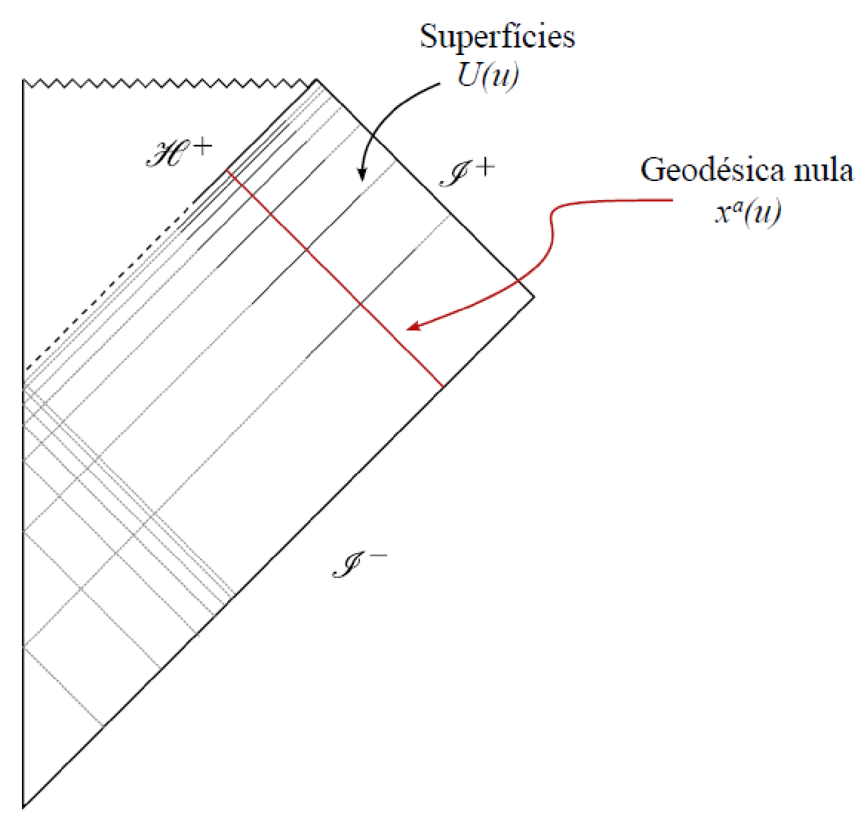
$$\Phi^{in+}_{\omega} = \int_0^{\infty} \left(A_{\omega\omega'} \Phi^{out+}_{\omega'} + B_{\omega\omega'} \Phi^{out*+}_{\omega'}\right) d\omega'$$

Relate the two, get B and from there is trivial to get the particle creation rate

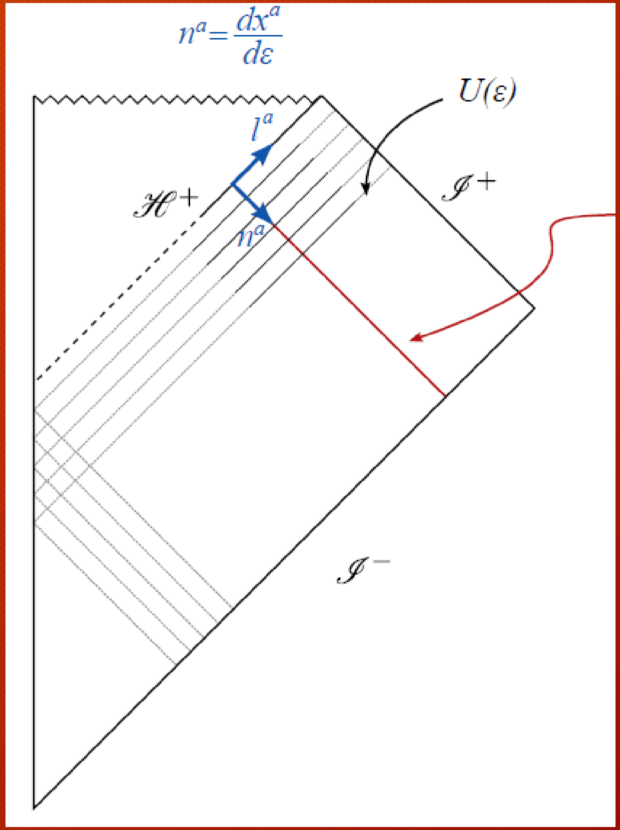
$$\Phi^{out-}_{\omega}(v) = \begin{cases} \Phi_0 \exp\left[-i\frac{\omega}{\kappa}\log(v_0 - v)\right] & \text{se } v \leq v_0 \\ 0 & \text{se } v > v_0 \end{cases}$$

**Result:** exponential relation between v and u coordinate parameters

# The basic idea …




vs




$$U\left(e^{-\kappa u}\right) \;=\; U\left(\frac{v_0 - v}{c}\right)$$

# End result

- Number of particles (scalar, massless) created in the future infinity with frequency omega:

$$N_\omega = \Gamma_\omega \frac{1}{\exp\left(\frac{2\pi\omega}{\kappa}\right) - 1}$$

$\Gamma_\omega$ can be interpreted as the BH cross section for raditation with frequency omega

- To be compared to

$$\langle n_\omega \rangle = \sum_{n=0}^{\infty} n P_\omega(n) = \frac{1}{\exp\frac{\omega}{T} - 1}$$

so $T = \frac{\kappa}{2\pi}$

Similar results can be obtained for other bosons and fermions, with or without mass

# Back reaction

- Stefan-Boltzmann law:

$$\frac{dM}{dt} = \sigma T^4 A$$

$$\frac{dM}{dt} = -\sigma \left(\frac{1}{8\pi M}\right)^4 \left[\pi (2M)^2\right] = -\frac{\sigma}{1024\pi^3}\frac{1}{M^2} \equiv -\frac{\lambda}{M^2}$$

- Black holes will evaporate (sigma~1):

$$t_{\mathrm{evap}} \sim 10^{76} \left(\frac{M}{M_{Sol}}\right)^3 \mathrm{s}$$

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